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BY

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ON THE OSCULATING QUARTIC OF A PLANE CURVE*

BY

WILLIAM WELLS DENTON

The quartic curve which has contact of the thirteenth order with a given analytic plane curve at one of its points will be called the osculating quartic. In this paper, its equation will be found in an invariant form, referred to a triangle which has a simple projective relation to the given curve. The method which we shall employ is due to Professor WILCZYNSKI.†

Let there be given three linearly independent analytic functions of x,

(1) $$y_\kappa = f_\kappa(x); \qquad (\kappa = 1, 2, 3).$$

They may be interpreted as the homogeneous coördinates of a point P_y in a plane. As x changes, P_y describes an analytic plane curve C_y. There exists a uniquely defined linear differential equation of the third order of which y_1, y_2, y_3 form a fundamental system. Let this be

(2) $$y^{(3)} + p_1 y'' + p_2 y' + p_3 y = 0,$$

where p_1, p_2, p_3 are analytic functions of x. We may, therefore, speak of C_y as being an integral curve of (2). But this integral curve is not unique, for every projective transformation of C_y is likewise an integral curve of (2). Conversely, every integral curve of (2) is a projective transformation of C_y. Hence the properties of C_y determined by the coefficients (2) are common to all curves projectively equivalent to C_y, that is, they are projective properties.

The representation of a curve in the form (1), however, involves some arbitrary elements. In the first place, since the coördinates are homogeneous, a transformation of the form

(3) $$\bar{y} = \lambda(x) y,$$

where $\lambda(x)$ is an arbitrary analytic function of x, does not affect the curve.

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† E. J. WILCZYNSKI, *Projective differential geometry of curves and ruled surfaces.* Teubner, Leipzig, 1906, Chapters II and III. We shall hereafter refer to this work as Proj. Diff. Geom.

Moreover, a transformation of the form

(4) $$\bar{x} = \xi(x),$$

where $\xi(x)$ is an arbitrary analytic function of x, merely changes the parametric representation, without changing the curve. Those combinations of the coefficients of (2) and of their derivatives, which remain unchanged under all transformations of the form (3) and (4), the so-called invariants, therefore characterize the projective properties of the curve independently of its method of representation. If such an invariant function contains also y and its derivatives it is called a covariant.

It is convenient to consider the subgroup of the general group of transformations defined by equations (3) and (4), which is obtained by leaving the independent variable x fixed. The invariant functions for this subgroup are known as seminvariants and semicovariants. Of these we shall need *

(5) $$z = y' + p_1 y, \qquad \rho = y'' + 2p_1 y' + p_2 y,$$

(6) $$P_2 = p_2 - p_1^2 - p_1', \qquad P_3 = p_3 - 3p_1 p_2 + 2p_1^3 - p_1'',$$

as well as the following invariants: †

(7) $$\theta_3 = P_3 - \tfrac{3}{2}P_2', \qquad \theta_8 = 6\theta_3\theta_3'' - 7(\theta_3')^2 - 27P_2\theta_3^2.$$

By a special transformation of the form (3) and (4), viz., ‡

(8) $$\bar{y} = \frac{d\bar{x}}{dx}y, \qquad \bar{x} = \xi(x),$$

where $\xi(x)$ is determined by the equations:

(9) $$\xi(x) = c_1\int e^{\int \eta dx}dx + c_2, \qquad \eta' - \tfrac{1}{2}\eta^2 = \tfrac{3}{2}P_2,$$

in which c_1 and c_2 are arbitrary constants, equation (2) may be reduced to the *Laguerre-Forsyth* canonical form:

(10) $$\frac{d^3\bar{y}}{d\bar{x}^3} + \bar{P}_3\bar{y} = 0;$$

so that the invariant functions above mentioned have the following exceptionally simple *canonical* forms:

(11) $$\bar{z} = y', \qquad \bar{\rho} = y'', \qquad \bar{P}_2 = 0,$$

(12) $$\bar{\theta}_3 = \bar{P}_3, \qquad \bar{\theta}_8 = 6\bar{P}_3\bar{P}_3'' - 7(\bar{P}_3')^2.$$

* *Proj. Diff. Geom.*, p. 58.

† Ibid., p. 59.

‡ Ibid., p. 25.

Hereafter, we shall assume the differential equation in this form, and, for convenience in writing, omit the dashes. We shall assume also that P_3, in this equation, is not identically equal to zero. This requires only that Cy be not a conic.

Let x_0 be the value of x which determines the point Py_0, on the curve Cy. The point Pz, whose coördinates are

$$z_\kappa = \left[\frac{dy_\kappa}{dx}\right]_{x=x_0} \qquad (\kappa=1, 2, 3),$$

is a point on the tangent to Cy at Py_0;§ and, if Py_0 is not a point of inflection, the point $P\rho$, whose coördinates are

$$\rho_\kappa = \left[\frac{d^2y_\kappa}{dx}\right]_{x=x_0} \qquad (\kappa=1, 2, 3),$$

is not collinear with Py_0 and Pz; so that these points determine a non-degenerate triangle, semicovariantly related to the curve Cy. Let this be taken as a triangle of reference. We may choose the unit point of our system of homogeneous coördinates so that an expression of the form ||

$$x_1 y(x_0) + x_2 z(x_0) + x_3 \rho(x_0)$$

will represent the point whose coördinates are precisely x_1, x_2, x_3.

Let $x = x_0$ be an ordinary point for the function P_3; and, for convenience in writing, let $x_0 = 0$, since this assumption involves no loss of generality. Then for values of $|x|$ sufficiently small, any ~~the~~ solution of equation (10) may be expressed as a convergent power series,

$$(14) \quad G(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \cdots + \frac{1}{14!}y^{(14)}(0)x^{14} + \cdots.$$

From equation (10), we find by direct differentiation:

$$
(15) \quad
\begin{aligned}
y^{(3)} &= -ay, \qquad y^{(4)} = -a_1 y - az, \\
y^{(5)} &= -a_2 y - 2a_1 z - a\rho, \\
y^{(6)} &= -(a_3 - a^2)y - 3a_2 z - 3a_1\rho, \text{ etc.,}
\end{aligned}
$$

where we have put $y \equiv y(0)$, $z \equiv y'(0)$, $\rho \equiv y''(0)$, $a \equiv P_3$, and where a_m^n is an abbreviation for the nth power of the mth derivative of P_3 with respect to x. Putting $G(x)$ in the form (13), we find the following equations, which represent the curve Cy up to terms of the fourteenth order in the vicinity of the point Py_0:

§ *Proj. Diff. Geom.*, p. 54.

|| *Proj. Diff. Geom.*, p. 61.

$$y_1 = 1 - \frac{a}{3!}x^3 - \frac{a_1}{4!}x^4 - \frac{a_2}{5!}x^5 - \frac{1}{6!}(a_3 - a^2)x^6 - \frac{1}{7!}(a_4 - 5aa_1)x^7$$

$$- \frac{1}{8!}(a_5 - 11aa_2 - 5a_1^2)x^8 - \frac{1}{9!}(a_6 - 21aa_3 - 21a_1 a_2 + a^3)x^9$$

$$- \frac{1}{10!}(a_7 - 36aa_4 - 42a_1 a_3 - 21a_2^2 + 12a^2 a_1)x^{10}$$

$$- \frac{1}{11!}(a_8 - 57aa_5 - 78a_1 a_4 - 84a_2 a_3 + 39a_2 a^2 + 45aa_1^2)x^{11}$$

$$- \frac{1}{12!}(a_9 - 85aa_6 - 135a_1 a_5 - 162a_2 a_4 + 105a^2 a_3 - 84a_3^2$$
$$+ 300aa_1 a_2 + 45a_1^3 - a^4)x^{12}$$

$$- \frac{1}{13!}(a_{10} - 220a_1 a_6 - 121aa_7 - 297a_2 a_5 - 330a_3 a_4 + 852aa_1 a_3$$
$$+ 246a^2 a_4 + 435a_1^2 a_2 + 516aa_2^2 - 22a^3 a_1)x^{13}$$

$$- \frac{1}{14!}(a_{11} - 166aa_8 - 341a_1 a_7 - 517a_2 a_6 - 627a_3 a_5 + 519a^2 a_5$$
$$- 330a_4^2 + 2124aa_1 a_4 + 1287a_1^2 a_3 + 3054aa_2 a_3$$
$$+ 1386a_1 a_2^2 - 94a^3 a_2 - 177a^2 a_1^2)x^{14}$$
$$- \cdots,$$

$$y_2 = x - \frac{a}{4!}x^4 - \frac{2a_1}{5!}x^5 - \frac{3a_2}{6!}x^6 - \frac{1}{7!}(4a_3 - a^2)x^7 - \frac{1}{8!}(5a_4 - 7aa_1)x^8$$

$$- \frac{1}{9!}(6a_5 - 18aa_2 - 12a_1^2)x^9 - \frac{1}{10!}(7a_6 + a^3 - 39aa_3 - 63a_1 a_2)x^{10}$$

$$- \frac{1}{11!}(8a_7 - 75aa_4 - 144a_1 a_3 - 84a_2^2 + 15a^2 a_1)x^{11}$$

$$- \frac{1}{12!}(9a_8 - 132aa_5 - 297a_1 a_4 - 396a_2 a_3 + 54a^2 a_2 + 75aa_1^2)x^{12} \tag{16}$$

$$- \frac{1}{13!}(10a_9 - 217aa_6 - 564a_1 a_5 - 855a_2 a_4 + 159a^2 a_3 - 480a_3^2$$
$$+ 558aa_1 a_2 + 120a_1^3 - a^4)x^{13}$$

$$- \frac{1}{14!}(11a_{10} - 338aa_7 - 1001a_1 a_6 - 1716a_2 a_5 - 2145a_3 a_4 + 405a^2 a_4$$
$$+ 1728aa_1 a_3 + 1353a_1^2 a_2 + 1074aa_2^2 - 26a^3 a_1)x^{14}$$
$$- \cdots,$$

$$y_3 = \frac{1}{2}x^2 - \frac{a}{5!}x^5 - \frac{3a_1}{6!}x^6 - \frac{6a_2}{7!}x^7 - \frac{1}{8!}(10a_3 - a^2)x^8$$

$$- \frac{1}{9!}(15a_4 - 9aa_1)x^9 - \frac{1}{10!}(21a_5 - 27aa_2 - 21a_1^2)x^{10}$$

$$-\frac{1}{11!}(28a_6 - 66aa_3 - 132a_1a_2 + a^3)x^{11}$$

$$-\frac{1}{12!}(36a_7 - 141aa_4 - 342a_1a_3 - 216a_2^2 + 18a^2a_1)x^{12}$$

$$-\frac{1}{13!}(45a_8 - 273aa_5 - 780a_1a_4 - 1170a_2a_3 + 72a_2a^2 + 111aa_1^2)x^{13}$$

$$-\frac{1}{14!}(55a_9 - 490aa_6 - 1617a_1a_5 - 2805a_2a_4 + 231a^2a_3$$
$$- 1650a_3^2 + 924aa_1a_2 + 231a_1^3 - a^4)x^{14}$$

$$- \cdots.$$

Let $Q(x_1, x_2, x_3) = 0$ be the equation of the osculating quartic. If we should substitute into its left member $x_\kappa = y_\kappa$ $(\kappa = 1, 2, 3)$, the coefficients of all powers of x up to and including the thirteenth would be equal to zero, and we should have fourteen equations for the ratios of the fifteen coefficients of Q. The equation $Q = 0$ has been obtained by the author in this way. The details of that solution, however, will not be given here, as the equation of the osculating quartic may be obtained more easily and in a more satisfactory form by another method. We wish, however, to emphasize the fact that the equations obtained by the two different methods have been compared, for the sake of checking the results presented here, and have been found to agree.

By combining y_1, y_2, y_3 so as to eliminate the terms in x up to the fourth and eighth orders, inclusive, Professor Wilczynski has obtained as the equations of the osculating conic and cubic respectively:*

(17) $$x_2^2 - 2x_1x_3 = 0,$$

(18) $$7(15P_3P_3^{(3)} - 20P_3'P_3'' - 567P_3^3)\Omega_1(x) + 20[6P_3P_3'' - 7(P_3')^2]\Omega_2(x) = 0,$$

where

(19) $$\Omega_1(x) = 5(x_2^2 - 2x_1x_3)(P_3'x_3 - 3P_3x_2) + 12P_3^2x_3^3,$$

(20) $$\Omega_2(x) = 5(x_2^2 - 2x_1x_3)(21P_3x_1 - P_3''x_3) - 42P_3^2x_2x_3^2 - 14P_3P_3'x_3^3.$$

The curve $\Omega_1(x) = 0$ has a special significance for our problem. It is the only cubic of the pencil of cubics having eight consecutive points in common with C_y, which has a double point at P_y, and has therefore been called the eight-pointic nodal cubic.

The work will be simplified if we introduce a system of non-homogeneous coördinates X, Y defined by the following equations:

* *Proj. Diff. Geom.*, pp. 63, 64.

$$t_1 = y_1 - \frac{a_1}{3a} y_2 + \frac{a_1^2}{18a^2} y_3,$$

$$(21) \qquad t_2 = \theta_1 y_2 - \frac{\theta_1 a_1}{3a} y_3,$$

$$t_3 = \theta_1^2 y_3,$$

$$(22) \qquad X = \frac{t_2}{t_1}, \qquad Y = \frac{t_3}{t_1},$$

in which θ_1 is an abbreviation for one of the cube roots of $-a/20$. The new triangle of reference, given by (21), is characterized by the fact that if Y be developed according to powers of X, the development assume the form:

$$(23) \qquad Y = \tfrac{1}{2}X^2 + X^5 + \alpha_7 X^7 + \cdots + \alpha_{14} X^{14} + \cdots,$$

in which all the coefficients are absolute invariants. Professor Wilczynski has shown that in order to obtain the canonical form (23) of the development, the triangle of reference must be chosen as follows:* "One vertex is a point on the curve and one side of the triangle is the tangent at this point. The second side is the line upon which are situated the three points of inflection of the eight-pointic nodal cubic. The third side is the polar of the intersection of the other two with respect to the osculating conic. The numerical factors, which still remain arbitrary in a projective system of coördinates after the triangle of reference has been chosen, must be determined in such a way that the coördinates of one of the three points of inflection of the eight-pointic nodal cubic shall be $(0, -\sqrt[3]{16}, 1)$, and that the coördinates of the tangent to the cubic at this point shall be $(2\sqrt[3]{16}, 3\sqrt[3]{16^2}, 48)$." We are thus dealing with a coördinate system which has a purely projective relation to the curve C_y.

We now proceed to obtain the development (23) explicitly up to terms of the fourteenth order. In addition to the two fundamental invariants θ_3 and θ_8, it will be necessary to have explicit expressions for the following system of invariants of equation (10), obtained from θ_3 and θ_8 by the Jacobian process:

$$\theta_{12} = 3\theta_3\theta_8' - 8\theta_8\theta_3', \qquad \theta_{16} = \theta_3\theta_{12}' - 4\theta_{12}\theta_3',$$

$$(24) \qquad \theta_{20} = 3\theta_3\theta_{16}' - 16\theta_{16}\theta_3', \qquad \theta_{24} = 3\theta_3\theta_{20}' - 20\theta_{20}\theta_3',$$

$$\theta_{28} = \theta_3\theta_{24}' - 8\theta_{24}\theta_3', \qquad \theta_{32} = 3\theta_3\theta_{28}' - 28\theta_{28}\theta_3',$$

From (12), we find:

* *Proj. Diff. Geom.*, p. 86.

$$
\begin{aligned}
\theta_{12} &= 2\cdot 3^2 a^2 a_3 - 2^3\cdot 3^2 a a_1 a_2 + 2^3\cdot 7 a_1^3,\\
\theta_{16} &= 2\cdot 3^2 a^3 a_4 - 2^2\cdot 3^3 a^2 a_1 a_3 - 2^3\cdot 3^2 a^2 a_2^2 + 2^7\cdot 3 a a_1^2 a_2 - 2^5\cdot 7 a_1^4,\\
\theta_{20} &= 2\cdot 3^3 a^4 a_5 - 2\cdot 3^2\cdot 5^2 a^3 a_1 a_4 + 2^3\cdot 3^2\cdot 31 a^2 a_1^2 a_3 - 2^2\cdot 3^3\cdot 7 a^3 a_2 a_3\\
&\quad + 2^4\cdot 3^3\cdot 7 a^2 a_1 a_2^2 - 2^9\cdot 3\cdot 5 a a_1^3 a_2 + 2^9\cdot 7 a_1^5,\\
\theta_{24} &= 2\cdot 3^4 a^5 a_6 - 2\cdot 3^4\cdot 11 a^4 a_1 a_5 - 2\cdot 3^3\cdot 67 a^4 a_2 a_4 + 2\cdot 3^2\cdot 647 a^3 a_1^2 a_4\\
&\quad - 2^2 3^4 7 a^4 a_3^2 + 2^2\cdot 3^5\cdot 41 a^3 a_1 a_2 a_3 - 2^4\cdot 3^2\cdot 13\cdot 29 a^2 a_1^3 a_3\\
&\quad - 2^5\cdot 3^4\cdot 43 a^2 a_1^2 a_2^2 + 2^{12}\cdot 3^2\cdot 5 a a_1^4 a_2 + 2^4\cdot 3^4\cdot 7 a^3 a_2^3 - 2^{11}\cdot 5\cdot 7 a_1^6,\\
\theta_{28} &= 2\cdot 3^4 a^6 a_7 - 2^2\cdot 3^4\cdot 7 a^5 a_1 a_6 + 2\cdot 3^2\cdot 7\cdot 149 a^4 a_1^2 a_5 - 2^3\cdot 3^3\cdot 5^2 a^5 a_2 a_5\\
&\quad + 2^4\cdot 3^2\cdot 7^2\cdot 11 a^4 a_1 a_2 a_4 - 2\cdot 3^3\cdot 151 a^5 a_3 a_4 - 2\cdot 3^2\cdot 7\cdot 893 a^3 a_1^3 a_4\\
&\quad + 2^2\cdot 3^4\cdot 151 a^4 a_1 a_3^2 - 2^2\cdot 3^2\cdot 16251 a^3 a_1^2 a_2 a_3 + 2^2\cdot 3^6\cdot 23 a^4 a_2^2 a_3 \qquad (25)\\
&\quad + 2^5\cdot 3^2\cdot 7\cdot 253 a^2 a_1^4 a_3 + 2^6\cdot 3^2\cdot 2441 a^2 a_1^3 a_2^2 - 2^4\cdot 3^6\cdot 23 a^3 a_1 a_2^3\\
&\quad - 2^{14}\cdot 3\cdot 5\cdot 7 a a_1^5 a_2 + 2^{14}\cdot 5\cdot 7 a_1^7,\\
\theta_{32} &= 2\cdot 3^5 a^7 a_8 - 2^3\cdot 3^4\cdot 13 a^6 a_1 a_7 + 2\cdot 3^3\cdot 7\cdot 227 a^5 a_1^2 a_6 - 2^2\cdot 3^4\cdot 71 a^6 a_2 a_6\\
&\quad - 2\cdot 3^2\cdot 7\cdot 5063 a^4 a_1^3 a_5 + 2^2\cdot 3^4\cdot 1283 a^5 a_1 a_2 a_5 - 2\cdot 3^4\cdot 251 a^6 a_3 a_5\\
&\quad - 2\cdot 3^2\cdot 222757 a^4 a_1^2 a_2 a_4 + 2^2\cdot 3^3\cdot 4019 a^5 a_2 a_4 + 2\cdot 3^3\cdot 11711 a^5 a_1 a_3 a_4\\
&\quad - 2\cdot 3^4\cdot 151 a^6 a_4^2 + 2\cdot 3^2\cdot 7\cdot 29111 a^3 a_1^4 a_4 - 2^2\cdot 3^5\cdot 7\cdot 373 a^4 a_1^2 a_3^2\\
&\quad + 2^2\cdot 3^5\cdot 5\cdot 113 a^5 a_2 a_3^2 + 2^2\cdot 3^3\cdot 237707 a^3 a_1^3 a_2 a_3\\
&\quad - 2^3\cdot 3^4\cdot 10799 a^4 a_1 a_2^2 a_3 - 2^6\cdot 3^2\cdot 7\cdot 4063 a^2 a_1^5 a_3 - 2^7\cdot 3^3\cdot 16417 a^2 a_1^4 a_2^2\\
&\quad + 2^4\cdot 3^4\cdot 13697 a^3 a_1^2 a_2^3 - 2^4\cdot 3^6\cdot 69 a^4 a_2^4 + 2^{19}\cdot 3\cdot 5\cdot 7 a a_1^6 a_2 - 2^{16}\cdot 5\cdot 7^2 a_1^8.
\end{aligned}
$$

We shall also need $\theta_{36} = 3\theta_3\theta'_{32} - 32\theta_{32}\theta'_3$. These Jacobians, together with θ_3 and θ_8, from a complete system (Σ) of invariants of equation (10), in the sense that any rational invariant whatever, involving $\theta_3 \equiv a$, and its derivatives, $\theta_3^{(i)} \equiv a_i\,(i = 1, 2 \cdots 9)$, may be expressed rationally in terms of the invariants of the system.* Equations (25) show that a, a_2, a_3, $\cdots$, a_9 may be expressed rationally in terms of the members of Σ and of a_1. Since the coefficients α_i in the development (23) are absolute invariants, when a and its derivatives have been replaced by the members of this system, the terms in α_i involving a_1 must in the aggregate disappear. We may therefore neglect, at the outset, all terms involving a_1; and it will be understood that terms involving a_1 have been omitted from the right hand members of all the "equations" which follow, up to and including (31). Equations (22) now take the simple form:

* *Proj. Diff. Geom.*, p. 36.

(26) $$X = \frac{\theta_1 y_2}{y_1}, \qquad Y = \frac{\theta_1^2 y_3}{y_1};$$

or more explicitly:

$$\theta_1^{-1} X = x + \frac{3}{4!} a x^4 + \frac{3}{6!} a_2 x^6 + \frac{3}{7!}(33a^2 + a_3) x^7 + \frac{3}{8!} a_4 x^8$$

$$+ \frac{3}{9!}(183 a a_2 + a_5) x^9 + \frac{3}{10!}(a_6 + 273 a a_3 + 3753 a^3) x^{10}$$

(27) $$+ \frac{3}{11!}(a_7 + 388 a a_4 + 413 a_2^2) x^{11} + \frac{3}{12!}(a_8 + 531 a a_5 + 1512 a_2 a_3$$

$$+ 60165 a^2 a_2) x^{12} + \frac{3}{13!}(a_9 + 705 a a_6 + 2586 a_2 a_4$$

$$+ 118377 a^2 a_3 + 1512 a_3^2 + 1017441 a^4) x^{13} + \cdots,$$

$$\theta_1^{-2} Y = \frac{1}{2} x^2 + \frac{9}{5!} a x^5 + \frac{15}{7!} a_2 x^7 + \frac{9}{8!}(2a_3 + 53a^2) x^8 + \frac{21}{9!} a_4 x^9$$

$$+ \frac{24}{10!}(a_5 + 150 a a_2) x^{10} + \frac{9}{11!}(3a_6 + 671 a a_3 + 8289 a^3) x^{11}$$

(28) $$+ \frac{3}{12!}(10 a_7 + 3171 a a_4 + 3570 a_2^2) x^{12} + \frac{3}{13!}(11 a_8 + 4758 a a_5$$

$$+ 14508 a_2 a_3 + 497772 a^2 a_2) x^{13} + \frac{3}{14!}(12 a_9 + 6867 a a_6$$

$$+ 27195 a_2 a_4 + 1069173 a^2 a_3 + 16020 a_3^2 + 8580087 a^4) X^{14} + \cdots.$$

Reverting the series (27), we have:

$$x = \theta_1^{-1} X - \frac{3}{4!} a \theta_1^{-4} X^4 - \frac{3}{6!} a_2 \theta_1^{-6} X^6 + \frac{3}{7!}(72a^2 - a_3)\theta_1^{-7} X^7$$

$$- \frac{3}{8!} a_4 \theta_1^{-8} X^8 + \frac{3}{9!}(447 a a_2 - a_5)\theta_1^{-9} X^9 + \frac{3}{10!}(717 a a_3 - 23058 a^3$$

(29) $$- a_6)\theta_1^{-10} X^{10} + \frac{3}{11!}(1097 a a_4 + 973 a_2^2 - a_7)\theta_1^{-11} X^{11} + \frac{3}{12!}(1614 a a_5$$

$$+ 3636 a_2 a_3 - 443691 a^2 a_2 - a_8)\theta_1^{-12} X^{12} + \frac{3}{13!}(2298 a a_6 + 6423 a_2 a_4$$

$$- \cancel{+}\, 985815 a^2 a_3 + 3636 a_3^2 + 18311440 a^4 - a_9)\theta_1^{-13} X^{13} + \cdots.$$

Substituting this value of x in equation (28), we find the following symbolic expression for the development (23):

$$
\begin{aligned}
Y = {} & \frac{1}{2}X^2 + X^5 - \frac{6}{7!}a_2\theta_1^{-5}X^7 + \frac{6}{8!}(105a^2 - a_3)\theta_1^{-6}X^8 - \frac{6}{9!}a_4\theta_1^{-7}X^9 \\
& + \frac{6}{10!}(630aa_2 - a_5)\theta_1^{-8}X^{10} + \frac{6}{11!}(990aa_3 - 43848a^3 - a_6)\theta_1^{-9}X^{11} \\
& + \frac{6}{12!}(1485aa_4 + 1386a_2^2 - a_7)\theta_1^{-10}X^{12} + \frac{6}{13!}(2145aa_5 + 5148a_2a_3 \\
& - 806922a^2a_2 - a_8)\theta_1^{-11}X^{13} + \frac{6}{14!}(3003a_6 + 9009a_2a_4 \\
& - 1741284a^2a_3 + 5148a_3^2 + 46540494a^4 - a_9)\theta_1^{-12}X^{14} + \cdots.
\end{aligned} \tag{30}
$$

It remains to replace the quantities a_i by the members of the system of invariants Σ of the differential equation (10). Solving equations (25) for these quantities, we find:

$$
\begin{aligned}
& a = \theta_3, \qquad 2\cdot 3\theta_3\cdot a_2 = \theta_8, \qquad 2\cdot 3^2\theta_3^2\cdot a_3 = \theta_{12}, \\
& 2\cdot 3^2\theta_3^3\cdot a_4 = \theta_{16} + 2\theta_8^2, \qquad 2\cdot 3^3\theta_3^4\cdot a_5 = \theta_{20} + 7\theta_{12}\theta_8, \\
& 2^2\cdot 3^4\theta_3^5\cdot a_6 = 2\theta_{24} + 67\theta_{16}\theta_8 + 2\cdot 7\theta_{12}^2 + 2\cdot 5^2\theta_8^3, \\
& 2^2\cdot 3^5\theta_3^6\cdot a_7 = 2\cdot 3\theta_{28} + 2^2\cdot 5^2\theta_{20}\theta_8 + 151\theta_{16}\theta_{12} + 3\cdot 127\theta_{12}\theta_8^2, \\
& 2^2\cdot 3^6\theta_3^7\cdot a_8 = 2\cdot 3\theta_{32} + 2\cdot 71\theta_{24}\theta_8 + 2\cdot 3\cdot 5^2\cdot 17\theta_{16}\theta_8^2 + 251\theta_{20}\theta_{12} \\
& \qquad + 3\cdot 151\theta_{16}^2 + 2^5\cdot 3\cdot 11\theta_{12}^2\theta_8 + 2\cdot 3\cdot 5^2\cdot 7\theta_8^4, \\
& 2^2\cdot 3^7\theta_3^8\cdot a_9 = 2\cdot 3\theta_{36} + 2\cdot 3\cdot 97\theta_{28}\theta_8 + 3\cdot 131\theta_{24}\theta_{12} + 3\cdot 7\cdot 79\theta_{20}\theta_{16} \\
& \qquad + 2\cdot 5^2\cdot 103\theta_{20}\theta_8^2 + 2\cdot 7681\theta_{16}\theta_{12}\theta_8 \\
& \qquad + 2^5\cdot 3\cdot 11\theta_{12}^3 + 2\cdot 3\cdot 2351\theta_{12}\theta_8^3,
\end{aligned} \tag{31}
$$

where we still neglect to write the terms involving a_1. The required coefficients α_i are therefore the following absolute invariants:

$$
\begin{aligned}
\alpha_7 &= 2^{-2}\cdot 3^{-2}\cdot 7^{-1}\theta_3^{-2}\theta_1^{-2}\theta_8, \\
\alpha_8 &= -2^{-3}\cdot 3^{-3}\cdot 5\cdot 7^{-1}\theta_3^{-4}(\theta_{12} - 2\cdot 3^3\cdot 5\cdot 7\theta_3^4), \\
\alpha_9 &= -2^{-3}\cdot 3^{-5}\cdot 5\cdot 7^{-1}\theta_3^{-5}\theta_1^{-1}(\theta_{16} + 2\theta_8^2), \\
\alpha_{10} &= -2^{-4}\cdot 3^{-6}\cdot 7^{-1}\theta_3^{-6}\theta_1^{-2}(\theta_{20} + 7\theta_{12}\theta_8 - 2\cdot 3^4\cdot 5\cdot 7\theta_8\theta_3^4), \\
\alpha_{11} &= 2^{-3}\cdot 3^{-7}\cdot 5\cdot 7^{-1}\cdot 11^{-1}\theta_3^{-8}(2\theta_{24} + 67\theta_{16}\theta_8 + 2\cdot 7\theta_{12}^2 - 2^2\cdot 3^4\cdot 5\cdot 11\theta_{12}\theta_3^4 \\
&\qquad + 2\cdot 5^2\theta_8^3 + 2^5\cdot 3^7\cdot 7\cdot 29\theta_3^8),
\end{aligned}
$$

$$
(32)\quad
\begin{aligned}
\alpha_{12} = {} & 2^{-5} \cdot 3^{-9} \cdot 5 \cdot 7^{-1} \cdot 11^{-1} \theta_3^{-9} \theta_1^{-1} (2 \cdot 3\theta_{28} + 2^2 \cdot 5^2 \theta_{20} \theta_8 + 151 \theta_{16} \theta_{12} \\
& + 3 \cdot 127 \theta_{12} \theta_8^2 - 2 \cdot 3^6 \cdot 5 \cdot 11 \theta_{16} \theta_3^4 - 2 \cdot 3^5 \cdot 11 \cdot 37 \theta_8^2 \theta_3^4), \\
\alpha_{13} = {} & 2^{-5} \cdot 3^{-10} \cdot 5 \cdot 7^{-1} \cdot 11^{-1} \cdot 13^{-1} \theta_3^{-10} \theta_1^{-2} (2 \cdot 3\theta_{32} + 2 \cdot 71 \theta_{24} \theta_8 + 251 \theta_{20} \theta_{12} \\
& - 2 \cdot 3^4 \cdot 5 \cdot 11 \cdot 13 \theta_{20} \theta_3^4 + 3 \cdot 151 \theta_{16}^2 + 2 \cdot 3 \cdot 5^2 \cdot 17 \theta_{16} \theta_8^2 + 2^5 \cdot 3 \cdot 11 \theta_{12}^2 \theta_8 \\
& - 2 \cdot 3^4 \cdot 11 \cdot 13 \cdot 41 \theta_{12} \theta_8 \theta_3^4 + 2 \cdot 3 \cdot 5^2 \cdot 7 \theta_8^4 + 2^2 \cdot 3^9 \cdot 17 \cdot 293 \theta_8 \theta_3^8); \\
\alpha_{14} = {} & - 2^{-4} \cdot 3^{-11} \cdot 5^2 \cdot 7^{-2} \cdot 11^{-1} \cdot 13^{-1} \theta_3^{-12} (2 \cdot 3\theta_{36} + 2 \cdot 3 \cdot 97 \theta_{28} \theta_8 + 3 \cdot 131 \theta_{24} \theta_{12} \\
& - 2 \cdot 3^4 \cdot 7 \cdot 11 \cdot 13 \theta_{24} \theta_3^4 + 3 \cdot 7 \cdot 79 \theta_{20} \theta_{16} + 2 \cdot 5^2 \cdot 103 \theta_{20} \theta_8^2 \\
& + 2 \cdot 7681 \theta_{16} \theta_{12} \theta_8 - 2^2 \cdot 3^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \theta_{16} \theta_8 \theta_3^4 + 2^5 \cdot 3 \cdot 11 \theta_{12}^3 \\
& - 2 \cdot 3^4 \cdot 5 \cdot 11^2 \cdot 13 \theta_{12}^2 \theta_3^4 + 2^3 \cdot 3^8 \cdot 16123 \theta_{12} \theta_3^8 + 2 \cdot 3 \cdot 2351 \theta_{12} \theta_8^3 \\
& - 2^2 \cdot 3^4 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \theta_8^3 \theta_3^4 - 2^3 \cdot 3^{11} \cdot 7^2 \cdot 11 \cdot 13 \cdot 41 \theta_3^{12}).
\end{aligned}
$$

Let the equation of the osculating quartic be assumed in the form:

$$
(33)\quad
\begin{aligned}
Q_Y = {} & \delta_{3333} Y^4 + \delta_{3332} Y^3 X + \delta_{3322} Y^2 X^2 + \delta_{3222} Y X^3 + \delta_{2222} X^4 + \delta_{333} Y^3 + \delta_{332} Y^2 X \\
& + \delta_{322} Y X^2 + \delta_{222} X^3 + \delta_{33} Y^2 + \delta_{32} Y X + \delta_{22} X^2 + \delta_3 Y + \delta_2 X + \delta = 0.
\end{aligned}
$$

If we substitute the development (23) for Y into the left member of this equation, the coefficients of all powers of X up to and including the thirteenth must be equal to zero, that is, we must have:

$$
(34)\quad
\begin{aligned}
& \delta = \delta_2 = 0, \qquad \delta_{22} + \tfrac{1}{2}\delta_3 = 0, \qquad \delta_{222} + \tfrac{1}{2}\delta_{32} = 0, \qquad \tfrac{1}{4}\delta_{33} + \tfrac{1}{2}\delta_{322} + \delta_{2222} = 0, \\
& \delta_{32} + \tfrac{1}{8}\delta_{333} + \tfrac{1}{4}\delta_{3322} = 0, \qquad \delta_3 + \tfrac{1}{4}\delta_{332} + \tfrac{1}{2}\delta_{3222} = 0, \\
& \alpha_7 \delta_3 + \delta_{33} + \delta_{322} + \tfrac{1}{8}\delta_{3332} = 0, \qquad \alpha_8 \delta_3 + \alpha_7 \delta_{32} + \delta_{332} + \delta_{3222} + \tfrac{1}{16}\delta_{3333} = 0, \\
& \alpha_9 \delta_3 + \alpha_8 \delta_{32} + \alpha_7 \delta_{33} + \alpha_7 \delta_{322} + \tfrac{3}{4}\delta_{333} + \delta_{3322} = 0, \\
& \alpha_{10} \delta_3 + \alpha_9 \delta_{32} + (1 + \alpha_8) \delta_{33} + \alpha_8 \delta_{322} + \alpha_7 \delta_{332} + \alpha_7 \delta_{3222} + \tfrac{3}{4}\delta_{3332} = 0, \\
& \alpha_{11}\delta_3 + \alpha_{10}\delta_{32} + \alpha_9\delta_{33} + \alpha_9\delta_{322} + (1 + \alpha_8)\delta_{332} + \alpha_8 \delta_{3222} + \alpha_7\delta_{3322} + \tfrac{3}{4}\alpha_7\delta_{333} + \tfrac{1}{2}\delta_{3333} = 0, \\
& \alpha_{12} \delta_3 + \alpha_{11} \delta_{32} + (\alpha_{10} + 2\alpha_7) \delta_{33} + \alpha_{10} \delta_{322} + \alpha_9 \delta_{332} + (\tfrac{3}{4}\alpha_8 + \tfrac{3}{2}) \delta_{333} \\
& \qquad + \alpha_9 \delta_{3222} + (1 + \alpha_8) \delta_{3322} + \tfrac{3}{4}\alpha_7 \delta_{3332} = 0, \\
& \alpha_{13} \delta_3 + \alpha_{12} \delta_{32} + (\alpha_{11} + 2\alpha_8) \delta_{33} + \alpha_{11} \delta_{322} + (\alpha_{10} + 2\alpha_7) \delta_{332} + \tfrac{3}{4}\alpha_9 \delta_{333} \\
& \qquad + \alpha_{10} \delta_{3222} + \alpha_9 \delta_{3322} + (\tfrac{3}{4}\alpha_8 + \tfrac{3}{2}) \delta_{3332} + \tfrac{1}{2}\alpha_7 \delta_{3333} = 0.
\end{aligned}
$$

Solving these equations, we find:

$$\delta_{22} = -2 \begin{vmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \end{vmatrix}, \qquad \delta_{222} = 2 \begin{vmatrix} d_{11} & d_{21} & d_{41} \\ d_{12} & d_{22} & d_{42} \\ d_{13} & d_{23} & d_{43} \end{vmatrix};$$

$$\text{(35)} \quad \delta_{33} - 4\delta_{2222} = 8 \begin{vmatrix} d_{11} & d_{31} & d_{41} \\ d_{12} & d_{32} & d_{42} \\ d_{13} & d_{33} & d_{43} \end{vmatrix}, \qquad \Delta_{2222} = 2 \begin{vmatrix} d_{21} & d_{31} & d_{41} \\ d_{22} & d_{32} & d_{42} \\ d_{23} & d_{33} & d_{43} \end{vmatrix},$$

$$\delta_{2222} = \begin{vmatrix} \alpha_7 & \alpha_8 - 4 & \alpha_9 & \alpha_{10} \\ d_{11} & d_{21} & d_{31} & d_{41} \\ d_{12} & d_{22} & d_{32} & d_{42} \\ d_{13} & d_{23} & d_{33} & d_{43} \end{vmatrix}, \qquad \delta_{332} = -4\Delta_{2222} - 32\delta_{22},$$

where

$$\text{(36)} \quad \begin{aligned} d_{11} &= \alpha_8 - 6, & d_{31} &= \alpha_{10} - \alpha_8\alpha_7 - 8\alpha_7, \\ d_{12} &= \alpha_9 - 2\alpha_7^2, & d_{32} &= \alpha_{11} - 2\alpha_9\alpha_7 - \alpha_8^2 - 4\alpha_8 + 12, \\ d_{13} &= \alpha_{10} - 2\alpha_8\alpha_7 - 4\alpha_7, & d_{33} &= \alpha_{12} - 3\alpha_8\alpha_9 - 8\alpha_7^2, \\ d_{21} &= \alpha_9 - \alpha_7^2, & d_{41} &= \alpha_{11} - \alpha_9\alpha_7 - 2\alpha_8 - 32; \\ d_{22} &= \alpha_{10} - 3\alpha_8\alpha_7 + 2\alpha_7, & d_{42} &= \alpha_{12} - 2\alpha_{10}\alpha_7 - \alpha_9\alpha_8 + 2\alpha_9 - 6\alpha_7^2, \\ d_{23} &= \alpha_{11} - \alpha_9\alpha_7 - 2\alpha_8^2 + 6\alpha_8 - 12, & d_{43} &= \alpha_{13} + 6\alpha_{10} - 2\alpha_{10}\alpha_8 - \alpha_9^2 - 14\alpha_8\alpha_7 - 28\alpha_7; \end{aligned}$$

so that the equation of the osculating quartic is:

$$\text{(37)} \quad \begin{aligned} &(\delta_{33} - 4\delta_{2222})(2Y - X^2 - 8XY^2)Y - \delta_{332}(X^3 + 16Y^3 - 2XY)Y \\ &\quad + 2\delta_{2222}(X^2 - 2Y)^2 + 2\delta_{222}(X^2 - 2Y)X + 2\delta_{22}(X^2 - 2Y) \\ &\quad + [2\alpha_7(\delta_{33} - 4\delta_{2222}) - 8\alpha_9\delta_{22} - 8(\alpha_8 - 4)\delta_{222}](X^2 - 2Y)Y^2 \\ &\quad + 64[(\alpha_8 - 2)\delta_{22} + \alpha_7\delta_{222}]Y^4 + 8(4\alpha_7\delta_{22}Y^2 + 2\delta_{222}XY + \delta_{22}X^2)XY = 0, \end{aligned}$$

where all of the coefficients are absolute invariants. This is easily reduced to homogeneous form, if desired, by equations (22). Referred to the same triangle, the equations of the osculating conic and cubic are respectively:

$$\text{(38)} \qquad X^2 - 2Y = 0,$$

$$\text{(39)} \quad \begin{aligned} &\alpha_7(2Y - X^2 - 8XY^2) + 2\alpha_7^2(X^2 - 2Y)Y \\ &\qquad\qquad + (\alpha_8 - 4)(X^3 + 16Y^3 - 2XY) = 0. \end{aligned}$$

When $\alpha_7 = 0$, (39) reduces to the equation of the eight-pointic nodal cubic.

If we substitute the canonical development (23) of the curve Cy, into the left member of the equation (33) of the osculating quartic, we find that the coefficient of X^{14} is

$$\alpha_{14}\delta_3+\alpha_{13}\delta_{32}+(\alpha_{12}+2\alpha_9+\alpha_7^2)\delta_{33}+\alpha_{12}\delta_{322}+(\alpha_{11}+2\alpha_8)\delta_{332}+(\tfrac{3}{4}\alpha_{10}+3\alpha_7)\delta_{333}$$
$$+\alpha_{11}\delta_{3222}+(\alpha_{10}+2\alpha_7)\delta_{3322}+\tfrac{3}{4}\alpha_9\delta_{3332}+(\tfrac{1}{2}\alpha_8+\tfrac{3}{2})\delta_{3333}.$$

When this expression vanishes, the osculating quartic has fifteen consecutive points in common with Cy at Py_0, or, as we may say, hyperosculates Cy at that point. This condition may be put into the form:

$$(40)\qquad K=\begin{vmatrix} d_{11} & d_{21} & d_{31} & d_{41} \\ d_{12} & d_{22} & d_{32} & d_{42} \\ d_{13} & d_{23} & d_{33} & d_{43} \\ d_{14} & d_{24} & d_{34} & d_{44} \end{vmatrix}=0,$$

where

$$(41)\qquad \begin{aligned} d_{14}&=\alpha_{11}-2\alpha_9\alpha_7-4\alpha_8-\alpha_7^3-24, \\ d_{24}&=\alpha_{12}-\alpha_{10}\alpha_7-2\alpha_9\alpha_8+6\alpha_9-\alpha_8\alpha_7^2-2\alpha_7^2, \\ d_{34}&=\alpha_{13}-\alpha_{10}\alpha_8-2\alpha_9^2-\alpha_9\alpha_7^2-16\alpha_8\alpha_7, \\ d_{44}&=\alpha_{14}+6\alpha_{11}-3\alpha_{10}\alpha_9-14\alpha_9\alpha_7-\alpha_{10}\alpha_7^2-8\alpha_8^2-40\alpha_8-144. \end{aligned}$$

If the invariant equation (40) is satisfied for all values of x, that is, at all points of the curve C_y, this curve is itself a quartic.

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